Uniqueness of positive bound states with multi-bump for nonlinear Schrödinger equations

Daomin Cao, Shuanglong Li, Peng Luo

Institute of Applied Mathematics, AMSS, The Chinese Academy of Sciences, Beijing 100190, PR China

Abstract

We are concerned with the following nonlinear Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u,\ u \in H^1(\mathbb{R}^N),$$

where $N \geq 3$, $2 < p < \frac{2N}{N-2}$. For ε small enough and a class of $V(x)$, we show the uniqueness of positive multi-bump solutions concentrating at k different critical points of $V(x)$ under certain assumptions on asymptotic behavior of $V(x)$ and its first derivatives near those points. The degeneracy of critical points is allowed in this paper.

Keywords: Nonlinear Schrödinger equations, Concentration, Uniqueness, Pohozaev identity

1. Introduction

In this paper, we consider the uniqueness of concentrating solutions to the following nonlinear Schrödinger equations

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u,\ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $N \geq 3$, $2 < p < \frac{2N}{N-2}$.

Problem (1.1) appears in the study of standing waves for the following nonlinear Schrödinger equations

$$i\varepsilon\frac{\partial\varphi}{\partial t} = -\varepsilon^2\Delta_x\varphi + (V(x)+E)\varphi - |\varphi|^{p-1}\varphi,\ (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{1.2}$$

with the form $\varphi(x,t) = e^{-iEt/\varepsilon}u(x)$, where i is the imaginary unit and ε is the Planck constant. Equation (1.2) is one of the most important problem in nonlinear optics and quantum physics. Especially, it describes the transition from quantum to classical mechanics as $\varepsilon \to 0$, so the study of solutions to (1.2) for small ε has a crucial interest in physical.

Email addresses: `dmcao@amt.ac.cn` (Daomin Cao), `lishuanglong@amss.ac.cn` (Shuanglong Li), `luopeng@whu.edu.cn` (Peng Luo)

In recent decades, there are a number of results on the existence of the solutions for problem (1.1). In [20], for the case $N = 1, p = 3$, Floer and Weinstein obtained a solution concentrating at the global non-degenerate minimum point when ε is small enough. Later, Oh [30, 31] generalized Floer-Weinstein's results to higher dimension with $2 < p < 2N/(N-2)$ and they obtained the existence of positive multi-bump solutions concentrating at any given set of nondegenerate critical points of $V(x)$ as $\varepsilon \to 0$. In fact, the results in [20, 30, 31] seem to rely in essential way on the nondegeneracy of the critical points. Also, the existence of a single-bump solution concentrating at the critical point of $V(x)$ which may be degenerate as $\varepsilon \to 0$ was obtained by Ambrosetti, Badiale, Cingolani [1]. These results were obtained by Lyapunov-Schmidt reduction.

On the other hand, by using variational methods, Rabinowitz [33] proved the existence of a positive ground solution to (1.1) for small ε if $V(x) \in C^1(\mathbb{R}^N)$ satisfies

$$\liminf_{|x|\to\infty} V(x) > \inf_{\mathbb{R}^N} V(x) > 0.$$

Later, Wang [35] gave the accurate characterization of the concentration behavior for the positive ground state and bound state solutions. The solutions in [33, 35] are mainly single-bump solutions. For the multi-bump solutions, del Pino and Felmer [15, 17] got the existence of such solutions concentrating at the critical points of $V(x)$ under some local conditions for the potential $V(x)$. Here they assumed that $V(x)$ is locally Hölder continuous on $\mathbb{R}^N$ and satisfies

$$\inf_{x\in\partial\Omega_i} V(x) > \inf_{x\in\Omega_i} V(x) > 0, \ i = 1, \cdots, k,$$

where $\Omega_1, \cdots, \Omega_k$ are k disjoint bounded regions. And the results in [17] are true when the critical points of $V(x)$ are degenerate.

There are also a lot of results concerning on the existence of multi-bump solutions for problems similar to (1.1). For Dirichlet problems with a subcritical nonlinearity on bounded domains, the solutions concentrating at one or a couple of points were obtained in [8, 28, 29]; For the case of a critical nonlinearity, the results on the existence of multi-bump solutions can be found in [4, 34]. For other results concerning the existence of the solutions with the concentration phenomena, one can refer to [2, 3, 5, 11, 14, 16, 19, 23, 24, 26] and the references therein.

As far as we know, the results on the uniqueness of solutions which have the concentration phenomena are few. In this aspect, the first result is the uniqueness of solutions concentrating at one point for Dirichlet problems with critical nonlinearity on bounded domains given by Glangetas in [22]. Later, Cao and Heinz [7] gained the uniqueness of solutions to (1.1) which concentrate at the nondegenerate critical points of $V(x)$. The results in [7, 22] are obtained by using the topological degree. Recently, Deng, Lin and Yan [18] got the local uniqueness and periodicity for the solutions with infinitely many bumps of the prescribed scalar curvature problem which involves the critical Sobolev exponent by the technique of Pohozaev identity. For more work concerning the uniqueness of solutions with the concentration phenomena, one can also refer to [9, 25].

However, whether the multi-bump solutions of problem (1.1) concentrating at the degenerate critical points of $V(x)$ are unique is still unknown. In this paper, inspired by [18] we solve this

problem partially by using the Pohozaev identity. To be specific, if two families of multi-bump solutions to (1.1) concentrate at the same set of critical points of $V(x)$, then they can be written in the same form by the results of [7]; next, we can get the useful estimate and use the local Pohozaev identity to show that the two solutions are in fact the same. Thus our uniqueness are essentially in the local sense.

Let $U_a(x)$ be the unique positive solution (see [27]) of the following problem

$$\begin{cases}-\Delta u+V(a)u=|u|^{p-2}u, \text{ in } \mathbb{R}^N,\\ u(0)=\max\limits_{x\in\mathbb{R}^N}u(x),\ u(x)\in H^1(\mathbb{R}^N),\end{cases}\tag{1.3}$$

where a is a given point in $\mathbb{R}^N$ and $V(a)>0$. Also it is well-known that $U_a(x)$ is a radially symmetric decreasing function satisfying for $|\alpha|\le 1$

$$|D^\alpha U_a(x)|\exp\big(\sqrt{V(a)}|x|\big)|x|^{\frac{N-1}{2}}\le C,\tag{1.4}$$

where $C>0$ is some constant in [21].

In our paper, we consider a class of $V(x)$ as follows:

(V_1): *$V(x)$ is a bounded C^1 function satisfying* $\inf\limits_{x\in\mathbb{R}^N}V(x)>0$.

(V_2): *$V(x)$ satisfies the following expansions:*

$$\begin{cases}V(x)=V(a_j)+b_j|x-a_j|^m+O(|x-a_j|^{m+1}), & x\in B_\delta(a_j),\\ \nabla V(x)=mb_j|x-a_j|^{m-2}(x-a_j)+O(|x-a_j|^m), & x\in B_\delta(a_j),\end{cases}\tag{1.5}$$

where $\delta>0$ is a small constant, $V(a_j)>0$, $m>1$, $b_j\neq 0$ and $j=1,\cdots,k$.

Next, we define

$$H_\varepsilon=\left\{u(x)\in H^1(\mathbb{R}^N),\int_{\mathbb{R}^N}\big(\varepsilon^2|\nabla u(x)|^2+V(x)u^2(x)\big)\mathrm{d}x<\infty\right\},$$

and for any $u(x)\in H_\varepsilon$,

$$\|u\|_\varepsilon=\big(u(x),u(x)\big)_\varepsilon^{\frac12}=\Big(\int_{\mathbb{R}^N}(\varepsilon^2|\nabla u(x)|^2+V(x)u^2(x))\mathrm{d}x\Big)^{\frac12}.$$

Then we have the following results:

Theorem 1.1. *Let $\{u_\varepsilon^{(1)}(x)\}_{\varepsilon>0},\{u_\varepsilon^{(2)}(x)\}_{\varepsilon>0}$ be two families of positive solutions of (1.1) concentrating at a set of k different points $\{a_1,\cdots,a_k\}\subset\mathbb{R}^N$. Suppose that $V(x)$ satisfies (V_1) and (V_2). Then for ε small enough, $u_\varepsilon^{(1)}(x)\equiv u_\varepsilon^{(2)}(x)$ and $u_\varepsilon(x)=u_\varepsilon^{(1)}(x)=u_\varepsilon^{(2)}(x)$ is of the form*

$$u_\varepsilon(x)=\sum_{j=1}^k U_{a_j}\Big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\Big)+w_\varepsilon(x),\tag{1.6}$$

with $x_{j,\varepsilon},w_\varepsilon(x)$ satisfying, for $j=1,\cdots,k$, as $\varepsilon\to0$,

$$|x_{j,\varepsilon}-a_j|=o(\varepsilon)\ \text{and}\ \|w_\varepsilon\|_\varepsilon=O(\varepsilon^{\frac N2+m}).\tag{1.7}$$

Remark 1.2. *Cao and Heinz [7] proved the uniqueness of solutions concentrating at the nondegenerate critical points of* $V(x)$ *by the topological degree. To using the topological degree in [7], it is assumed that the critical points set* $\{a_1, \cdots, a_k\}$ *of* $V(x)$ *are nondegenerate and* $V(x)$ *is* C^2 *at* $\{a_1, \cdots, a_k\}$*. However, Theorem 1.1 shows the uniqueness of solutions concentrating at the critical points of* $V(x)$ *which may be degenerate under the conditions* (V_1) *and* (V_2)*. We point out that even under the same assumptions as in [7], the proofs in the present paper are much simpler than those in [7].*

For the case $m = 2$ *in* (1.5)*, suppose that* $\{a_1, \cdots, a_k\}$ *are the nondegenerate critical points of* $V(x)$*, then Theorem 1.1 is the same as the results in [7]. However, the framework of using the topological degree in [7, 22] does not work anymore for the case* $m \neq 2$ *in* (1.5)*. Here, we use the Pohozaev identity to prove our main results.*

Specifically, if $m > 2$ *in* (1.5)*, then* $\{a_1, \cdots, a_k\}$ *are the degenerate critical points of* $V(x)$ *and our results show the uniqueness of solutions concentrating at the degenerate critical points.*

If $1 < m < 2$ *in* (1.5)*,* $V(x)$ *is not* C^2 *at the critical points set* $\{a_1, \cdots, a_k\}$*. And we can also obtain the uniqueness of solutions concentrating at* $\{a_1, \cdots, a_k\}$*. There, it should point out that we need new and careful estimates to hand this case.*

These mean that our results extend the results in [7] to more general cases which include the degenerate case.

Remark 1.3. *Since problem* (1.1) *is the case of a subcritical nonlinearity and the positive solution of* (1.3) *can not be given explicitly, different arguments from [18] should be applied to obtain the estimates we need in our proof of Theorem 1.1. Another point should be pointed out is that the interaction between the bumps must be taken into careful consideration.*

Remark 1.4. *The role of Pohozaev identity in the existence and nonexistence of solutions to problems with critical Sobolev exponents has been showed in many papers, see [6, 12, 13, 32] and the references therein. But the role of Pohozaev identity in the uniqueness is not well exploited until recently (see [18, 25]), we expect more applications of it in the further.*

Our paper is organized as follows. In Section 2, we obtain some estimates which are essential to prove Theorem 1.1. Next, by using the Pohozaev identity we give the detailed proof of Theorem 1.1 in Section 3. Finally, we give estimates of some important quantities used repeatedly in this paper and their proofs in the Appendix.

Throughout this paper, we will use the same C to denote various generic positive constants, and we will use $O(t)$, $o(t)$ to mean $|O(t)| \leq C|t|$, $o(t)/t \to 0$ as $t \to 0$. Finally, $o(1)$ denotes quantities that tend to 0 as $\varepsilon \to 0$.

2. Preliminaries

First, we define for any $a, y \in \mathbb{R}^N$

$$E_{\varepsilon,a,y} = \left\{ u(x) \in H^1(\mathbb{R}^N) : \big(u(x), U_a(\frac{x-y}{\varepsilon})\big)_\varepsilon = 0,\ \big(u(x), \frac{\partial U_a(\frac{x-y}{\varepsilon})}{\partial x_i}\big)_\varepsilon = 0,\ i = 1, \cdots, N \right\}.$$

Then, the following basic structure of the solutions concentrating at k different points has been obtained by the Lyapunov-Schmidt reduction in [7].

Proposition 2.1. *If $\{u_\varepsilon(x)\}_{\varepsilon>0}$ is a family of positive solutions of* (1.1) *concentrating at a set of k different points $\{a_1,\cdots,a_k\}$, then $a_j (j=1,\cdots,k)$ must be a critical point of $V(x)$ and $u_\varepsilon(x)$ is of the form*

$$u_\varepsilon(x)=\sum_{j=1}^{k}(1+\alpha_{j,\varepsilon})U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon})+\upsilon_\varepsilon(x), \tag{2.1}$$

with $x_{j,\varepsilon}$, $\upsilon_\varepsilon(x)\in\bigcap_{j=1}^{k}E_{\varepsilon,a_j,x_{j,\varepsilon}}$ satisfying, for $j=1,\cdots,k$, as $\varepsilon\to 0$,

$$|x_{j,\varepsilon}-a_j|=o(1),\ \ \alpha_{j,\varepsilon}=o(1),\ \ \|\upsilon_\varepsilon\|_\varepsilon=o(\varepsilon^{\frac{N}{2}}). \tag{2.2}$$

Proof. See [[7] Theorem 1.1]. □

Remark 2.2. *Proposition 2.1 shows that if a family of solutions has concentration phenomenon with multi-bump, then the solutions can be written in the form* (2.1)*. Also, letting*

$$w_\varepsilon(x)=\sum_{j=1}^{k}\alpha_{j,\varepsilon}U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon})+\upsilon_\varepsilon(x)$$

and using (2.2)*, we can write the solution $u_\varepsilon(x)$ in Proposition 2.1 in the following form:*

$$u_\varepsilon(x)=\sum_{j=1}^{k}U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon})+w_\varepsilon(x), \tag{2.3}$$

with $x_{j,\varepsilon}$, $w_\varepsilon(x)$ satisfying, for $j=1,\cdots,k$, as $\varepsilon\to 0$,

$$|x_{j,\varepsilon}-a_j|=o(1),\ \ \|w_\varepsilon\|_\varepsilon=o(\varepsilon^{\frac{N}{2}}). \tag{2.4}$$

In this paper, for simplicity, we will use (2.1) *and* (2.3) *alternately.*

Next, by the regularity theory of elliptic equations, $u_\varepsilon(x)$ is in fact a classical solution. Then, we establish the Pohozaev identity which is crucial in our paper.

Proposition 2.3. *Let $u(x)$ be the solution of* (1.1)*, then for any bounded open domain Ω, we have the following Pohozaev identity*

$$\begin{aligned}\int_\Omega\frac{\partial V(x)}{\partial x_i}u^2(x)\mathrm{d}x=&-2\varepsilon^2\int_{\partial\Omega}\frac{\partial u(x)}{\partial\nu}\frac{\partial u(x)}{\partial x_i}\mathrm{d}\sigma+\varepsilon^2\int_{\partial\Omega}|\nabla u(x)|^2\nu_i(x)\mathrm{d}\sigma\\&+\int_{\partial\Omega}V(x)u^2(x)\nu_i(x)\mathrm{d}\sigma-\frac{2}{p}\int_{\partial\Omega}|u(x)|^p\nu_i(x)\mathrm{d}\sigma,\end{aligned} \tag{2.5}$$

where $\nu(x)=\big(\nu_1(x),\cdots,\nu_N(x)\big)$ is the outward unit normal of $\partial\Omega$ and $i\in\{1,\cdots,N\}$.

Proof. Multiplying $\frac{\partial u(x)}{\partial x_i}$ on both sides of (1.1) and integrating on Ω, we have

$$-\varepsilon^2\int_\Omega \Delta u(x)\frac{\partial u(x)}{\partial x_i}\mathrm{d}x+\int_\Omega V(x)u(x)\frac{\partial u(x)}{\partial x_i}\mathrm{d}x=\int_\Omega \frac{\partial u(x)}{\partial x_i}|u(x)|^{p-2}u(x)\mathrm{d}x. \tag{2.6}$$

Next,

$$\begin{aligned}\text{LHS of (2.6)}=&-\varepsilon^2\int_{\partial\Omega}\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial \nu}\mathrm{d}\sigma+\varepsilon^2\int_\Omega \nabla u(x)\cdot\nabla\frac{\partial u(x)}{\partial x_i}\mathrm{d}x\\&+\frac{1}{2}\int_{\partial\Omega}u^2(x)V(x)\nu_i(x)\mathrm{d}\sigma-\frac{1}{2}\int_\Omega u^2(x)\frac{\partial V(x)}{\partial x_i}\mathrm{d}x,\end{aligned} \tag{2.7}$$

also,

$$\int_\Omega \nabla u(x)\cdot\nabla\frac{\partial u(x)}{\partial x_i}\mathrm{d}x=\frac{1}{2}\int_{\partial\Omega}|\nabla u(x)|^2\nu_i(x)\mathrm{d}\sigma. \tag{2.8}$$

On the other hand, by Green's formula, we have

$$\text{RHS of (2.6)}=\frac{1}{p}\int_{\partial\Omega}|u(x)|^p\nu_i(x)\mathrm{d}\sigma. \tag{2.9}$$

Then (2.5) follows from (2.6), (2.7), (2.8) and (2.9). □

In the rest of this section, we will show that the estimates of $|x_{j,\varepsilon}-a_j|$, $\alpha_{j,\varepsilon}$ and $\|w_\varepsilon\|_\varepsilon$ in Proposition 2.1 can be improved step by step.

Lemma 2.4. *Let $u_\varepsilon(x)$ be the solution of* (1.1) *with the form* (2.1)*. Suppose that* (V_1) *and* (V_2) *are satisfied, then we have*

$$|x_{j,\varepsilon}-a_j|=O(\varepsilon)+O(\sum_{l=1}^k\alpha_{l,\varepsilon}^{\frac{2}{m-1}}),\ \ j=1,\cdots,k. \tag{2.10}$$

Proof. First, taking $u(x)=u_\varepsilon(x)=\sum_{l=1}^k U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon})+w_\varepsilon(x)$ and $\Omega=B_d(x_{j,\varepsilon})$ for some small constant $d>0$ in the Pohozaev identity (2.5), we have, for $i=1,\cdots,N$,

$$\int_{B_d(x_{j,\varepsilon})}\frac{\partial V(x)}{\partial x_i}\big[\sum_{l=1}^k U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon})+w_\varepsilon(x)\big]^2\mathrm{d}x=I_1+I_2+I_3, \tag{2.11}$$

where

$$I_1=-2\varepsilon^2\int_{\partial B_d(x_{j,\varepsilon})}\frac{\partial\big(\sum_{l=1}^k U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon})+w_\varepsilon(x)\big)}{\partial\nu}\frac{\partial\big(\sum_{l=1}^k U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon})+w_\varepsilon(x)\big)}{\partial x_i}\mathrm{d}\sigma,$$

$$I_2=\int_{\partial B_d(x_{j,\varepsilon})}\big[\varepsilon^2\big|\nabla\big(\sum_{l=1}^k U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon})+w_\varepsilon(x)\big)\big|^2+V(x)\big(\sum_{l=1}^k U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon})+w_\varepsilon(x)\big)^2\big]\nu_i(x)\mathrm{d}\sigma,$$

and

$$I_3 = -\frac{2}{p}\int_{\partial B_d(x_{j,\varepsilon})} \big|\big(\sum_{l=1}^{k} U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon}) + w_\varepsilon(x)\big)\big|^p \nu_i(x)\mathrm{d}\sigma.$$

Now, using (A.1) and (A.3) in the Appendix, we have, for any $\gamma > 0$,

$$\begin{aligned}
&\int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x_i}\big[\sum_{l=1}^{k} U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon}) + w_\varepsilon(x)\big]^2 \mathrm{d}x \\
&\qquad = \int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x_i}\big[\sum_{l=1}^{k} U^2_{a_l}\big(\frac{x-x_{l,\varepsilon}}{\varepsilon}\big) + w^2_\varepsilon(x)\big]\mathrm{d}x + O(\varepsilon^\gamma) \\
&\qquad\quad + 2\sum_{l=1}^{k}\int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x_i} U_{a_l}\big(\frac{x-x_{l,\varepsilon}}{\varepsilon}\big) w_\varepsilon(x)\mathrm{d}x \\
&\qquad = \int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x_i} U^2_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)\mathrm{d}x + O(\|w_\varepsilon\|^2_\varepsilon + \varepsilon^\gamma) \\
&\qquad\quad + 2\int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x_i} U_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big) w_\varepsilon(x)\mathrm{d}x.
\end{aligned} \tag{2.12}$$

So, by choosing $\gamma > 0$ appropriately, from (2.12) and (A.6), we have

$$\begin{aligned}
&\int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x_i}\big[\sum_{l=1}^{k} U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon}) + w_\varepsilon(x)\big]^2 \mathrm{d}x \\
&= \int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x_i} U^2_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)\mathrm{d}x + O\big(\|w_\varepsilon\|^2_\varepsilon + \varepsilon^{N+2m-2} + \varepsilon^N |x_{j,\varepsilon} - a_j|^{2m-2}\big).
\end{aligned} \tag{2.13}$$

Next, from (A.2) and Lemma A.5, we know, for any $\gamma > 0$,

$$I_1 = O\big(\|w_\varepsilon\|^2_\varepsilon + \varepsilon^\gamma\big),\ I_2 = O\big(\|w_\varepsilon\|^2_\varepsilon + \varepsilon^\gamma\big) \text{ and } I_3 = O\big(\int_{\partial B_d(x_{j,\varepsilon})} |w_\varepsilon(x)|^p \mathrm{d}\sigma + \varepsilon^\gamma\big). \tag{2.14}$$

Also, from (2.4), (A.11) and Lemma A.5, we obtain

$$\int_{\partial B_d(x_{j,\varepsilon})} |w_\varepsilon(x)|^p \mathrm{d}\sigma \le C\int_{\mathbb{R}^N} |w_\varepsilon(x)|^p \mathrm{d}x \le C\|w_\varepsilon\|^2_\varepsilon. \tag{2.15}$$

Then combining (2.14) and (2.15), we see that for any $\gamma > 0$,

$$I_1 + I_2 + I_3 = O\big(\|w_\varepsilon\|^2_\varepsilon + \varepsilon^\gamma\big). \tag{2.16}$$

From (2.4), (2.11), (2.13), (2.16) and (B.3), taking γ appropriately, for $i = 1, \cdots, N$, we have

$$\int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x_i} U^2_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)\mathrm{d}x = O\big(\varepsilon^N\big(\varepsilon^{2m-2} + \max_{l=1,\cdots,k} |x_{l,\varepsilon} - a_l|^{2m-2} + \sum_{l=1}^{N} \alpha^2_{l,\varepsilon}\big)\big). \tag{2.17}$$

On the other hand,

$$
\begin{aligned}
&\int_{B_d(x_{j,\varepsilon})}\frac{\partial V(x)}{\partial x_i}U_{a_j}^2(\frac{x-x_{j,\varepsilon}}{\varepsilon})\mathrm{d}x\\
&\qquad= b_j m\varepsilon^N\int_{B_{\frac{d}{\varepsilon}}(0)}|\varepsilon y+x_{j,\varepsilon}-a_j|^{m-2}\cdot(\varepsilon y_i+x_{j,\varepsilon,i}-a_{j,i})U_{a_j}^2(y)\mathrm{d}y\\
&\qquad\quad+O\big(\varepsilon^N\int_{B_{\frac{d}{\varepsilon}}(0)}|\varepsilon y+x_{j,\varepsilon}-a_j|^{m}U_{a_j}^2(y)\mathrm{d}y\big),
\end{aligned}
\tag{2.18}
$$

where y_i, $x_{j,\varepsilon,i}$, $a_{j,i}$ are the i-th components of y, $x_{j,\varepsilon}$, a_j for $i=1,\cdots,N$. And

$$
\int_{B_{\frac{d}{\varepsilon}}(0)}|\varepsilon y+x_{j,\varepsilon}-a_j|^{m}U_{a_j}^2(y)\mathrm{d}y=O(\varepsilon^m+|x_{j,\varepsilon}-a_j|^m). \tag{2.19}
$$

Then for $i=1,\cdots,N$, (2.17), (2.18) and (2.19) imply

$$
\begin{aligned}
&\Big|\int_{B_{\frac{d}{\varepsilon}}(0)}|\varepsilon y+x_{j,\varepsilon}-a_j|^{m-2}\cdot(\varepsilon y_i+x_{j,\varepsilon,i}-a_{j,i})U_{a_j}^2(y)\mathrm{d}y\Big|\\
&\qquad\le C\big(\varepsilon^{2m-2}+\varepsilon^m+|x_{j,\varepsilon}-a_j|^m+\max_{l=1,\cdots,k}|x_{l,\varepsilon}-a_l|^{2m-2}+\sum_{l=1}^N\alpha_{l,\varepsilon}^2\big).
\end{aligned}
\tag{2.20}
$$

Also, we have the following inequality:

$$
\big||a+b|^m-|a|^m-m|a|^{m-2}a\cdot b\big|\le C\big(|a|^{m-m^*}|b|^{m^*}+b^m\big), \tag{2.21}
$$

where $a=(a_1,\cdots,a_N)\in\mathbb{R}^N$, $b=(b_1,\cdots,b_N)\in\mathbb{R}^N$, $a\cdot b=\sum_{j=1}^N a_jb_j$, $m>1$, $m^*=\min\{m,2\}$ and the constant C is independent of a,b.

Letting $a=\varepsilon y+x_{j,\varepsilon}-a_j$ and $b=-\varepsilon y$ in (2.21), we can get

$$
\begin{aligned}
&\sum_{i=1}^N m|\varepsilon y+x_{j,\varepsilon}-a_j|^{m-2}(x_{j,\varepsilon,i}-a_{j,i})(\varepsilon y_i+x_{j,\varepsilon,i}-a_{j,i})\\
&\ge|x_{j,\varepsilon}-a_j|^m+(m-1)|\varepsilon y+x_{j,\varepsilon}-a_j|^m-C(|\varepsilon y|^m+|\varepsilon y|^{m^*}|x_{j,\varepsilon}-a_j|^{m-m^*})\\
&\ge|x_{j,\varepsilon}-a_j|^m-C(|\varepsilon y|^m+|\varepsilon y|^{m^*}|x_{j,\varepsilon}-a_j|^{m-m^*}).
\end{aligned}
\tag{2.22}
$$

Then multiplying $U_{a_j}^2(y)$ on both sides of (2.22) and integrating on $B_{\frac{d}{\varepsilon}}(0)$, we obtain

$$
\begin{aligned}
&|x_{j,\varepsilon}-a_j|\sum_{i=1}^N\Big|\int_{B_{\frac{d}{\varepsilon}}(0)}|\varepsilon y+x_{j,\varepsilon}-a_j|^{m-2}(\varepsilon y_i+x_{j,\varepsilon,i}-a_{j,i})U_{a_j}^2(y)\mathrm{d}y\Big|\\
&\ge\sum_{i=1}^N\big[(x_{j,\varepsilon,i}-a_{j,i})\int_{B_{\frac{d}{\varepsilon}}(0)}|\varepsilon y+x_{j,\varepsilon}-a_j|^{m-2}(\varepsilon y_i+x_{j,\varepsilon,i}-a_{j,i})U_{a_j}^2(y)\mathrm{d}y\big]\\
&\ge\frac{1}{m}|x_{j,\varepsilon}-a_j|^m\int_{B_{\frac{d}{\varepsilon}}(0)}U_{a_j}^2(y)\mathrm{d}y-\frac{C}{m}\int_{B_{\frac{d}{\varepsilon}}(0)}\big(|\varepsilon y|^m+|\varepsilon y|^{m^*}|x_{j,\varepsilon}-a_j|^{m-m^*}\big)U_{a_j}^2(y)\mathrm{d}y.
\end{aligned}
\tag{2.23}
$$

So (2.20) and (2.23) imply

$$
\begin{aligned}
|x_{j,\varepsilon}-a_j|^m \le C\big[&|x_{j,\varepsilon}-a_j|\big(\varepsilon^{2m-2}+\varepsilon^m+\sum_{l=1}^N \alpha_{l,\varepsilon}^2\big)+\max_{l=1,\cdots,k}|x_{l,\varepsilon}-a_l|^{2m-1} \\
&+|x_{j,\varepsilon}-a_j|^{m+1}+\varepsilon^m+|x_{j,\varepsilon}-a_j|^{m-m^*}\varepsilon^{m^*}\big].
\end{aligned}
\tag{2.24}
$$

Also, by Hölder's inequality, for any $\eta>0$, we know

$$
\begin{aligned}
&|x_{j,\varepsilon}-a_j|\varepsilon^{2m-2}+|x_{j,\varepsilon}-a_j|\sum_{l=1}^N \alpha_{l,\varepsilon}^2+|x_{j,\varepsilon}-a_j|^{m-m^*}\varepsilon^{m^*} \\
&\le \eta|x_{j,\varepsilon}-a_j|^m+C_\eta\big(\varepsilon^{2m}+\sum_{l=1}^N \alpha_{l,\varepsilon}^{\frac{2m}{m-1}}+\varepsilon^m\big).
\end{aligned}
\tag{2.25}
$$

Then combining (2.4), (2.24) and (2.25), we have

$$
|x_{j,\varepsilon}-a_j|^m \le C\big(\varepsilon^m+\sum_{l=1}^N \alpha_{l,\varepsilon}^{\frac{2m}{m-1}}\big).
$$

This means that (2.10) is true. □

Here we need (2.21) to handle the term $|\varepsilon y+x_{j,\varepsilon}-a_j|^{m-2}(\varepsilon y_i+x_{j,\varepsilon,i}-a_{j,i})$ in (2.20) because the index m may be less than 2, which is quite different from the technique in [18].

Next, Lemma 2.4 and Proposition B.2 imply

$$
\|w_\varepsilon\|_\varepsilon^2=O(\varepsilon^{N+2m})+O(\varepsilon^N\sum_{j=1}^k \alpha_{j,\varepsilon}^2),\quad \|v_\varepsilon\|_\varepsilon^2=O(\varepsilon^{N+2m})+O(\varepsilon^N\sum_{j=1}^k \alpha_{j,\varepsilon}^2).
\tag{2.26}
$$

Proposition 2.5. *Let $u_\varepsilon(x)$ be the solution of* (1.1) *with the form* (2.1), *suppose that* (V_1) *and* (V_2) *are satisfied, then we have*

$$
\alpha_{j,\varepsilon}=O(\varepsilon^m),\ j=1,\cdots,k.
\tag{2.27}
$$

Proof. From Proposition 2.1, let

$$
u_\varepsilon(x)=\sum_{j=1}^k(1+\alpha_{j,\varepsilon})U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon})+v_\varepsilon(x)
$$

be a positive solution of (1.1) concentrating at $\{a_1,\cdots,a_k\}$, then

$$
\|u_\varepsilon\|_\varepsilon^2=\int_{\mathbb{R}^N} u_\varepsilon^p(x)\mathrm{d}x.
\tag{2.28}
$$

Now we set

$$\bar{u}_\varepsilon(x) = \frac{u_\varepsilon(x)}{1+\alpha_{1,\varepsilon}} = \sum_{j=1}^{k} \beta_{j,\varepsilon} U_{a_j}\left(\frac{x-x_{j,\varepsilon}}{\varepsilon}\right) + \beta_\varepsilon v_\varepsilon(x), \tag{2.29}$$

where

$$\beta_{j,\varepsilon} = \frac{1+\alpha_{j,\varepsilon}}{1+\alpha_{1,\varepsilon}},\ \beta_\varepsilon = \frac{1}{1+\alpha_{1,\varepsilon}},\ j=1,\cdots,k. \tag{2.30}$$

Then (2.2) and (2.30) implies

$$\beta_{1,\varepsilon} = 1,\ \beta_{j,\varepsilon} = 1+o(1),\ \beta_\varepsilon = 1+o(1),\ j=2,\cdots,k. \tag{2.31}$$

Next, (2.28) and (2.29) show that

$$(1+\alpha_{1,\varepsilon})^{p-2} = \frac{\|\bar{u}_\varepsilon\|_\varepsilon^2}{\int_{\mathbb{R}^N} \bar{u}_\varepsilon^p(x)\mathrm{d}x}. \tag{2.32}$$

On the other hand, letting

$$K_\varepsilon(u) = \frac{\|u\|_\varepsilon^2}{\left(\int_{\mathbb{R}^N} u^p(x)\mathrm{d}x\right)^{\frac{2}{p}}},$$

by Lagrange multiplier method we can verify that $\bar{u}_\varepsilon(x)$ is a critical point of the functional $K_\varepsilon(u)$. Then from $DK_\varepsilon\big(\bar{u}_\varepsilon(x)\big)\big(U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\big)=0$, using the fact that $v_\varepsilon(x)$ and $U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon})$ are orthogonal, we get

$$\frac{\|\bar{u}_\varepsilon\|_\varepsilon^2}{\int_{\mathbb{R}^N} \bar{u}_\varepsilon^p(x)\mathrm{d}x} = \frac{\left(\sum_{j=1}^k \beta_{j,\varepsilon} U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon}), U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\right)_\varepsilon}{\int_{\mathbb{R}^N} |\bar{u}_\varepsilon(x)|^{p-2}\bar{u}_\varepsilon(x) U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x}. \tag{2.33}$$

It is not difficult to show the following inequality:

$$\Big|(\sum_{i=1}^{k+1} d_i)^l - d_1^l\Big| \le C\big(d_1^{l-1}|\sum_{i=2}^{k+1} d_i| + \sum_{i=2}^{k+1}|d_i|^l\big), \tag{2.34}$$

where $l>1$ and $d_i \in \mathbb{R}$.

Then from the fact that $\beta_{1,\varepsilon}=1$ in (2.31) and (2.34), we have

$$\begin{aligned} |\bar{u}_\varepsilon(x)|^{p-2}\bar{u}_\varepsilon(x) = {} & U_{a_1}^{p-1}\left(\frac{x-x_{1,\varepsilon}}{\varepsilon}\right) + O\Big(\sum_{j=2}^{k} U_{a_j}^{p-1}\left(\frac{x-x_{j,\varepsilon}}{\varepsilon}\right) + |v_\varepsilon(x)|^{p-1}\Big) \\ & + O\Big(U_{a_1}^{p-2}\left(\frac{x-x_{1,\varepsilon}}{\varepsilon}\right)\big(\sum_{j=2}^{k} U_{a_j}\left(\frac{x-x_{j,\varepsilon}}{\varepsilon}\right) + |v_\varepsilon(x)|\big)\Big). \end{aligned} \tag{2.35}$$

Then combining (2.35) and (A.3), we can deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\bar{u}_\varepsilon(x)|^{p-2}\bar{u}_\varepsilon(x)U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} U_{a_1}^p(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x + O\big(\int_{\mathbb{R}^N} U_{a_1}^{p-1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})|\upsilon_\varepsilon(x)|\mathrm{d}x\big) \\
&\qquad + O\big(\int_{\mathbb{R}^N} |\upsilon_\varepsilon(x)|^{p-1}U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x\big) + O(\varepsilon^\gamma).
\end{aligned} \tag{2.36}$$

Next, (1.3), (2.10), (2.26) and (A.5) imply

$$\begin{aligned}
\int_{\mathbb{R}^N} U_{a_1}^{p-1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})|\upsilon_\varepsilon(x)|\mathrm{d}x &= \int_{\mathbb{R}^N} \big(V(a_1)-V(x)\big)U_{a_1}\big(\frac{x-x_{1,\varepsilon}}{\varepsilon}\big)|\upsilon_\varepsilon(x)|\mathrm{d}x \\
&= O(\varepsilon^{N+2m} + \varepsilon^N\sum_{j=1}^k \alpha_{j,\varepsilon}^2).
\end{aligned} \tag{2.37}$$

Also, by Hölder's inequality and (2.26), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\upsilon_\varepsilon(x)|^{p-1}U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x \\
&\quad \le \big(\int_{\mathbb{R}^N} |\upsilon_\varepsilon(x)|^{2^*}\mathrm{d}x\big)^{\frac{p-1}{2^*}} \cdot \big(\int_{\mathbb{R}^N} U_{a_1}^{\frac{2^*}{2^*-p+1}}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x\big)^{1-\frac{p-1}{2^*}} \\
&\quad \le C\big(\varepsilon^{-1}\|\upsilon_\varepsilon\|_\varepsilon\big)^{p-1}\cdot\varepsilon^{N-\frac{(N-2)(p-1)}{2}} \\
&\quad \le C\big(\varepsilon^{N+m(p-1)} + \varepsilon^N\sum_{j=1}^k \alpha_{j,\varepsilon}^{p-1}\big).
\end{aligned} \tag{2.38}$$

Letting $p^* = \min\{p, 3\}$, from (2.36), (2.37) and (2.38), we see

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\bar{u}_\varepsilon(x)|^{p-2}\bar{u}_\varepsilon(x)U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} U_{a_1}^p(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x + O\big[\varepsilon^N(\varepsilon^{m(p^*-1)} + \sum_{j=1}^k \alpha_{j,\varepsilon}^{p^*-1})\big] \\
&\quad = \varepsilon^N\big[\int_{\mathbb{R}^N} U_{a_1}^p(x)\mathrm{d}x + O(\varepsilon^{m(p^*-1)} + \sum_{j=1}^k \alpha_{j,\varepsilon}^{p^*-1})\big].
\end{aligned} \tag{2.39}$$

Also from the fact that $\beta_{1,\varepsilon}=1$ in (2.31) and (A.3), we know

$$\begin{aligned}\Big(\sum_{j=1}^{k}\beta_{j,\varepsilon}U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon}),U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\Big)_\varepsilon &= \Big(U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon}),U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\Big)_\varepsilon+O(\varepsilon^\gamma)\\ &=\int_{\mathbb{R}^N}\big(V(x)-V(a_1)\big)U_{a_1}^2(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}U_{a_1}^p(\frac{x-x_{1,\varepsilon}}{\varepsilon})\mathrm{d}x+O(\varepsilon^\gamma).\end{aligned}\tag{2.40}$$

So, from (2.10), (2.40) and (A.5), taking suitable $\gamma>0$, we obtain

$$\Big(\sum_{j=1}^{k}\beta_{j,\varepsilon}U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon}),U_{a_1}(\frac{x-x_{1,\varepsilon}}{\varepsilon})\Big)_\varepsilon=\varepsilon^N\int_{\mathbb{R}^N}U_{a_1}^p(x)\mathrm{d}x+O\big(\varepsilon^N\sum_{j=1}^{k}\alpha_{j,\varepsilon}^2+\varepsilon^{N+m}\big).\tag{2.41}$$

Then combining (2.32), (2.33), (2.39) and (2.41), we have

$$\begin{aligned}(1+\alpha_{1,\varepsilon})^{p-2}&=\frac{\int_{\mathbb{R}^N}U_{a_1}^p(x)\mathrm{d}x+O\big(\sum_{j=1}^k\alpha_{j,\varepsilon}^2\big)+O(\varepsilon^m)}{\int_{\mathbb{R}^N}U_{a_1}^p(x)\mathrm{d}x+O(\varepsilon^{m(p^*-1)}+\sum_{j=1}^k\alpha_{j,\varepsilon}^{p^*-1})}\\ &=1+O\big(\sum_{j=1}^{k}\alpha_{j,\varepsilon}^{p^*-1}\big)+O(\varepsilon^m).\end{aligned}$$

Similar to the above procedure, we can get

$$(1+\alpha_{i,\varepsilon})^{p-2}=1+O\big(\sum_{j=1}^{k}\alpha_{j,\varepsilon}^{p^*-1}\big)+O(\varepsilon^m),\ \text{for all } i=1,\cdots,k.\tag{2.42}$$

Also, by Taylor expansion, we have

$$(1+\alpha_{i,\varepsilon})^{p-2}=1+(p-2)\alpha_{i,\varepsilon}+o(\alpha_{i,\varepsilon}),\ \text{for all } i=1,\cdots,k.\tag{2.43}$$

Then (2.42) and (2.43) deduce

$$\alpha_{i,\varepsilon}=O\big(\sum_{j=1}^{k}\alpha_{j,\varepsilon}^{p^*-1}\big)+O(\varepsilon^m),\ \text{for all } i=1,\cdots,k.\tag{2.44}$$

Using the fact $p^*>2$ and summing (2.44) from $i=1$ to k, we obtain (2.27). □

Proposition 2.6. *Let w_ε be as in* (2.3)*, suppose that* (V_1) *and* (V_2) *are satisfied, then we have*

$$\|w_\varepsilon\|_\varepsilon=O(\varepsilon^{m+\frac{N}{2}}).\tag{2.45}$$

Proof. It is obvious that (2.26) and (2.27) imply (2.45). □

Furthermore, in next section, we need the following precise estimates about $|x_{j,\varepsilon} - a_j|$.

Proposition 2.7. *Let* $u_\varepsilon(x)$ *be the solution of* (1.1) *with the form* (2.1)*, suppose that* (V_1) *and* (V_2) *are satisfied, then we have*

$$|x_{j,\varepsilon} - a_j| = o(\varepsilon),\ \ j = 1, \cdots, k.$$

Proof. From Lemma 2.4 and Proposition 2.5, we have

$$|x_{j,\varepsilon} - a_j| = O(\varepsilon). \tag{2.46}$$

Then from (2.19) and (2.46), we obtain

$$\int_{B_{\frac{d}{\varepsilon}}(0)} \varepsilon^N |\varepsilon y + x_{j,\varepsilon} - a_j|^m U_{a_j}^2(y)\mathrm{d}y = O(\varepsilon^{N+m}). \tag{2.47}$$

So (2.17), (2.18) and (2.47) imply

$$\int_{B_{\frac{d}{\varepsilon}}(0)} |y + \frac{x_{j,\varepsilon} - a_j}{\varepsilon}|^{m-2} \cdot (y_i + \frac{x_{j,\varepsilon,i} - a_{j,i}}{\varepsilon}) U_{a_j}^2(y)\mathrm{d}y = O(\varepsilon) + O(\varepsilon^{m-1}), \tag{2.48}$$

where y_i, $x_{j,\varepsilon,i}$, $a_{j,i}$ are the i-th components of y, $x_{j,\varepsilon}$, a_j for $i = 1, \cdots, N$.

By choosing a subsequence, we can suppose that $\frac{x_{j,\varepsilon} - a_j}{\varepsilon} \to x_0$. Then letting $\varepsilon \to 0$ in (2.48), we have

$$\int_{\mathbb{R}^N} |y + x_0|^{m-2} \cdot (y_i + x_{0,i}) U_{a_j}^2(y)\mathrm{d}y = 0,$$

where $x_{0,i}$ is the i-th component of x_0 for $i = 1, \cdots, N$.

By the strictly decreasing of $U_{a_j}(x)$, we get $x_0 = 0$. That is $|x_{j,\varepsilon} - a_j| = o(\varepsilon)$. $\square$

3. Proof of the Main Theorem

Suppose that $u_\varepsilon^{(1)}(x)$, $u_\varepsilon^{(2)}(x)$ are two different positive solutions concentrating at $\{a_1, \cdots, a_k\}$, and

$$\xi_\varepsilon(x) = \frac{u_\varepsilon^{(1)}(x) - u_\varepsilon^{(2)}(x)}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\mathbb{R}^N)}}. \tag{3.1}$$

Then $\xi_\varepsilon(x)$ satisfies $\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 1$ and

$$-\varepsilon^2 \Delta \xi_\varepsilon(x) + V(x)\xi_\varepsilon(x) = C_\varepsilon(x)\xi_\varepsilon(x), \tag{3.2}$$

where

$$C_\varepsilon(x) = (p-1)\int_0^1 \big(tu_\varepsilon^{(1)}(x) + (1-t)u_\varepsilon^{(2)}(x)\big)^{p-2}\mathrm{d}t.$$

Proposition 3.1. *For $\xi_\varepsilon(x)$ defined by* (3.1), *we have*

$$\|\xi_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N}{2}}). \tag{3.3}$$

Proof. From (3.2) we have

$$\|\xi_\varepsilon\|_\varepsilon^2 = \int_{\mathbb{R}^N} C_\varepsilon(x)\xi_\varepsilon^2(x)\mathrm{d}x. \tag{3.4}$$

On the other hand,

$$|C_\varepsilon(x)| \le C\Big(\sum_{j=1}^{k} U_{a_j}^{p-2}\big(\frac{x-x_{j,\varepsilon}^{(1)}}{\varepsilon}\big)+\sum_{j=1}^{k} U_{a_j}^{p-2}\big(\frac{x-x_{j,\varepsilon}^{(2)}}{\varepsilon}\big)+|w_\varepsilon^{(1)}(x)|^{p-2}+|w_\varepsilon^{(2)}(x)|^{p-2}\Big). \tag{3.5}$$

Since $|\xi_\varepsilon(x)|\le 1$ and (2.4). For $l=1,2$, we know

$$\int_{\mathbb{R}^N} U_{a_j}^{p-2}\big(\frac{x-x_{j,\varepsilon}^{(l)}}{\varepsilon}\big)\xi_\varepsilon^2(x)\mathrm{d}x \le C\varepsilon^N, \tag{3.6}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} \big|w_\varepsilon^{(l)}(x)\big|^{p-2}\xi_\varepsilon^2(x)\mathrm{d}x &\le \Big(\int_{\mathbb{R}^N} \big|w_\varepsilon^{(l)}(x)\big|^{2^*}\mathrm{d}x\Big)^{\frac{p-2}{2^*}}\cdot\Big(\int_{\mathbb{R}^N}\big|\xi_\varepsilon(x)\big|^{\frac{2\cdot 2^*}{2^*-(p-2)}}\mathrm{d}x\Big)^{1-\frac{p-2}{2^*}}\\ &\le C\varepsilon^{(p-2)(\frac{N}{2}-1)}\|\xi_\varepsilon\|_\varepsilon^{2-\frac{(N-2)(p-2)}{N}}.\end{aligned} \tag{3.7}$$

Thus (3.4), (3.5), (3.6) and (3.7) imply

$$\|\xi_\varepsilon\|_\varepsilon^2 \le C\big(\varepsilon^N + \varepsilon^{(p-2)(\frac{N}{2}-1)}\|\xi_\varepsilon\|_\varepsilon^{2-\frac{(N-2)(p-2)}{N}}\big).$$

This leads to (3.3). ☐

Lemma 3.2. *Let $\xi_{\varepsilon,j}(x)=\xi_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)})$, then taking a subsequence necessarily, it holds*

$$\xi_{\varepsilon,j}(x) \to \sum_{i=1}^{N} b_{j,i}\psi_i(x)$$

uniformly in $C^1(B_R(0))$ for any $R>0$, where $b_{j,i}$, $i=1,\cdots,N$ are some constants and

$$\psi_i(x) = \frac{\partial U_{a_j}(x)}{\partial x_i},\ i=1,\cdots,N.$$

Proof. In view of $|\xi_{\varepsilon,j}(x)|\le 1$, we may assume that $\xi_{\varepsilon,j}(x)\to\xi_j(x)$ in $C_{loc}(\mathbb{R}^N)$. By direct calculations, we have

$$-\Delta\xi_{\varepsilon,j}(x) = -\varepsilon^2\Delta\xi_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)}) = -V(\varepsilon x + x_{j,\varepsilon}^{(1)})\xi_{\varepsilon,j}(x) + C_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)})\xi_{\varepsilon,j}(x). \tag{3.8}$$

Now, we estimate $C_\varepsilon(\varepsilon x + x^{(1)}_{j,\varepsilon})$,

$$U_{a_s}\big(\frac{x-x^{(1)}_{s,\varepsilon}}{\varepsilon}\big)-U_{a_s}\big(\frac{x-x^{(2)}_{s,\varepsilon}}{\varepsilon}\big)=O\big(\frac{x^{(1)}_{s,\varepsilon}-x^{(2)}_{s,\varepsilon}}{\varepsilon}\nabla U_{a_s}(\frac{x-x^{(1)}_{s,\varepsilon}}{\varepsilon})\big)=o(1)\nabla U_{a_s}\big(\frac{x-x^{(1)}_{s,\varepsilon}}{\varepsilon}\big),$$

for $s=1,\cdots,k$. Then

$$u^{(1)}_\varepsilon(x)-u^{(2)}_\varepsilon(x)=o(1)\sum_{s=1}^{k}\nabla U_{a_s}\big(\frac{x-x^{(1)}_{s,\varepsilon}}{\varepsilon}\big)+O\big(|w^{(1)}_\varepsilon(x)|+|w^{(2)}_\varepsilon(x)|\big). \tag{3.9}$$

So, from (A.1), for any $\gamma>0$ and $x\in B_d(x^{(1)}_{j,\varepsilon})$, we have

$$C_\varepsilon(x)=(p-1)U^{p-2}_{a_j}\big(\frac{x-x^{(1)}_{j,\varepsilon}}{\varepsilon}\big)+\big(o(1)\nabla U_{a_j}\big(\frac{x-x^{(1)}_{j,\varepsilon}}{\varepsilon}\big)+O\big(|w^{(1)}_\varepsilon(x)|+|w^{(2)}_\varepsilon(x)|\big)\big)^{p-2}+o(\varepsilon^\gamma).$$

Then we have

$$C_\varepsilon(\varepsilon x+x^{(1)}_{j,\varepsilon})=(p-1)U^{p-2}_{a_j}(x)+O\big(|w^{(1)}_\varepsilon(\varepsilon x+x^{(1)}_{j,\varepsilon})|+|w^{(2)}_\varepsilon(\varepsilon x+x^{(1)}_{j,\varepsilon})|\big)^{p-2}+o(1),\ x\in B_{\frac{d}{\varepsilon}}(0).$$

Next, for any given $\Phi(x)\in C^\infty_0(\mathbb{R}^N)$,

$$\begin{aligned}&\int_{\mathbb{R}^N}\big(-\Delta\xi_{\varepsilon,j}(x)+V(\varepsilon x+x^{(1)}_{j,\varepsilon})\xi_{\varepsilon,j}(x)-(p-1)U^{p-2}_{a_j}(x)\xi_{\varepsilon,j}(x)\big)\Phi(x)\mathrm{d}x\\&\qquad=O\big(\int_{\mathbb{R}^N}\big(|w^{(1)}_\varepsilon(\varepsilon x+x^{(1)}_{j,\varepsilon})|^{p-2}+|w^{(2)}_\varepsilon(\varepsilon x+x^{(1)}_{j,\varepsilon})|^{p-2}\big)|\Phi(x)|\mathrm{d}x\big)+o(1).\end{aligned}$$

Also, for $l=1,2$, we know

$$\begin{aligned}\int_{\mathbb{R}^N}|w^{(l)}_\varepsilon(\varepsilon x+x^{(1)}_{j,\varepsilon})|^{p-2}|\Phi(x)|\mathrm{d}x&\le C\big(\int_{\mathbb{R}^N}|w^{(l)}_\varepsilon(\varepsilon x+x^{(1)}_{j,\varepsilon})|^{2^*}\mathrm{d}x\big)^{\frac{p-2}{2^*}}\\&\le C\big(\varepsilon^{-1}\|w^{(l)}_\varepsilon\|_\varepsilon\big)^{p-2}\varepsilon^{-\frac{N(p-2)}{2^*}}\\&=C\big(\varepsilon^{-\frac{N}{2}}\|w^{(l)}_\varepsilon\|_\varepsilon\big)^{p-2}.\end{aligned} \tag{3.10}$$

Then using (2.4) and (3.10), we can obtain

$$\int_{\mathbb{R}^N}\big(-\Delta\xi_{\varepsilon,j}(x)+V(\varepsilon x+x^{(1)}_{j,\varepsilon})\xi_{\varepsilon,j}(x)-(p-1)U^{p-2}_{a_j}(x)\xi_{\varepsilon,j}(x)\big)\Phi(x)\mathrm{d}x=o(1). \tag{3.11}$$

Letting $\varepsilon\to 0$ in (3.11) and using the elliptic regularity theory, we find that $\xi_j(x)$ satisfies

$$-\Delta\xi_j(x)+V(a_j)\xi_j(x)=(p-1)U^{p-2}_{a_j}(x)\xi_j(x),\quad \text{in } \mathbb{R}^N,$$

which gives

$$\xi_j(x)=\sum_{i=1}^{N}b_{j,i}\psi_i(x).$$

□

Lemma 3.3. *Let $b_{j,i}$ be as in Lemma 3.2, then we have*

$$b_{j,i}=0,\quad \textit{for all } j=1,\cdots,k,\ i=1,\cdots N.$$

Proof. From Proposition 2.6, Proposition 3.1 and Lemma A.5, for some small δ, we have

$$\Big(\int_{\partial B_\delta(x^{(1)}_{j,\varepsilon})}\big(\varepsilon^2|\nabla w_\varepsilon(x)|^2+V(x)w^2_\varepsilon(x)\big)\mathrm{d}\sigma\Big)^{\frac{1}{2}}=O(\varepsilon^{\frac{N}{2}+m}),\tag{3.12}$$

and

$$\Big(\int_{\partial B_\delta(x^{(1)}_{j,\varepsilon})}\big(\varepsilon^2|\nabla w_\varepsilon(x)|^2+V(x)w^2_\varepsilon(x)\big)\mathrm{d}\sigma\Big)^{\frac{1}{2}}=O(\varepsilon^{\frac{N}{2}}).\tag{3.13}$$

Since $u^{(1)}_\varepsilon(x)$, $u^{(2)}_\varepsilon(x)$ are the positive solutions of (1.1), using Pohozaev identity (2.5), we deduce

$$\begin{aligned}&\int_{B_\delta(x^{(1)}_{j,\varepsilon})}\frac{\partial V(x)}{\partial x_i}\big(u^{(1)}_\varepsilon(x)+u^{(2)}_\varepsilon(x)\big)\cdot\xi_\varepsilon(x)\mathrm{d}x\\&=-2\varepsilon^2\int_{\partial B_\delta(x^{(1)}_{j,\varepsilon})}\Big(\frac{\partial\xi_\varepsilon(x)}{\partial\nu}\frac{\partial u^{(1)}_\varepsilon(x)}{\partial x_i}+\frac{\partial\xi_\varepsilon(x)}{\partial x_i}\frac{\partial u^{(2)}_\varepsilon(x)}{\partial\nu}\Big)\mathrm{d}\sigma-2\int_{\partial B_\delta(x^{(1)}_{j,\varepsilon})}A_\varepsilon(x)\xi_\varepsilon(x)\nu_i(x)\mathrm{d}\sigma\\&\quad+\int_{\partial B_\delta(x^{(1)}_{j,\varepsilon})}\big[\varepsilon^2\langle\nabla\big(u^{(1)}_\varepsilon(x)+u^{(2)}_\varepsilon(x)\big),\nabla\xi_\varepsilon(x)\rangle+V(x)\langle u^{(1)}_\varepsilon(x)+u^{(2)}_\varepsilon(x),\xi_\varepsilon(x)\rangle\big]\nu_i(x)\mathrm{d}\sigma,\end{aligned}\tag{3.14}$$

where

$$A_\varepsilon(x)=\int_0^1\big(tu^{(1)}_\varepsilon(x)+(1-t)u^{(2)}_\varepsilon(x)\big)^{p-1}\mathrm{d}t.$$

Then from (3.12), (3.13), (A.3) and (A.4), we get

$$\text{RHS of (3.14)}=O(\varepsilon^{N+m})-2\int_{\partial B_\delta(x^{(1)}_{j,\varepsilon})}A_\varepsilon(x)\xi_\varepsilon(x)\nu_i(x)\mathrm{d}\sigma.\tag{3.15}$$

Also, using Proposition 2.7 and (A.2), we have, for any $\gamma>0$,

$$|A_\varepsilon(x)|\le o(\varepsilon^\gamma)+\big(|w^{(1)}_\varepsilon(x)|+|w^{(2)}_\varepsilon(x)|\big)^{p-1},\ x\in\partial B_\delta(x^{(1)}_{j,\varepsilon}).$$

Then from (2.4), (A.11) and Lemma A.5, taking $\gamma>0$ appropriately, we get

$$\begin{aligned}\Big|\int_{\partial B_\delta(x^{(1)}_{j,\varepsilon})}A_\varepsilon(x)\xi_\varepsilon(x)\nu_i(x)\mathrm{d}\sigma\Big|&\le C\Big(\int_{\mathbb{R}^N}(|w^{(1)}_\varepsilon(x)|+|w^{(2)}_\varepsilon(x)|)^{p-1}|\xi_\varepsilon(x)|\mathrm{d}x+\varepsilon^\gamma\Big)\\&\le C\big((\|w^{(1)}_\varepsilon\|_\varepsilon+\|w^{(2)}_\varepsilon\|_\varepsilon)\|\xi_\varepsilon\|_\varepsilon+\varepsilon^\gamma\big)\le C\varepsilon^{N+m}.\end{aligned}\tag{3.16}$$

From (3.15) and (3.16), we know

$$\text{RHS of (3.14)}=O(\varepsilon^{N+m}).\tag{3.17}$$

On the other hand,

$$\begin{aligned}
&\int_{B_\delta(x_{j,\varepsilon}^{(1)})} \frac{\partial V(x)}{\partial x_i}\big(u_\varepsilon^{(1)}(x)+u_\varepsilon^{(2)}(x)\big)\xi_\varepsilon(x)\mathrm{d}x \\
&\qquad = mb_i\int_{B_\delta(x_{j,\varepsilon}^{(1)})}|x-a_j|^{m-2}(x_i-a_{j,i})\big(u_\varepsilon^{(1)}(x)+u_\varepsilon^{(2)}(x)\big)\xi_\varepsilon(x)\mathrm{d}x \\
&\qquad\quad + O\Big(\int_{B_\delta(x_{j,\varepsilon}^{(1)})}|x-a_j|^{m}\big(u_\varepsilon^{(1)}(x)+u_\varepsilon^{(2)}(x)\big)\xi_\varepsilon(x)\mathrm{d}x\Big).
\end{aligned} \tag{3.18}$$

From (3.9), we know

$$\begin{aligned}
u_\varepsilon^{(1)}(x)+u_\varepsilon^{(2)}(x) = {} & 2\sum_{s=1}^{k} U_{a_s}\big(\frac{x-x_{s,\varepsilon}^{(1)}}{\varepsilon}\big)+o(1)\sum_{s=1}^{k}\nabla U_{a_s}\big(\frac{x-x_{s,\varepsilon}^{(1)}}{\varepsilon}\big) \\
& + O\big(|w_\varepsilon^{(1)}(x)|+|w_\varepsilon^{(2)}(x)|\big).
\end{aligned} \tag{3.19}$$

Also, from (3.19) and (A.1), we can get

$$\begin{aligned}
&\int_{B_\delta(x_{j,\varepsilon}^{(1)})}|x-a_j|^{m-2}(x_i-a_{j,i})\big(u_\varepsilon^{(1)}(x)+u_\varepsilon^{(2)}(x)\big)\xi_\varepsilon(x)\mathrm{d}x \\
&\qquad = 2\int_{B_\delta(x_{j,\varepsilon}^{(1)})}|x-a_j|^{m-2}(x_i-a_{j,i})U_{a_j}\big(\frac{x-x_{j,\varepsilon}^{(1)}}{\varepsilon}\big)\xi_\varepsilon(x)\mathrm{d}x \\
&\qquad\quad + o(1)\int_{B_\delta(x_{j,\varepsilon}^{(1)})}|x-a_j|^{m-2}(x_i-a_{j,i})\nabla U_{a_j}\big(\frac{x-x_{j,\varepsilon}^{(1)}}{\varepsilon}\big)\xi_\varepsilon(x)\mathrm{d}x \\
&\qquad\quad + O\Big(\int_{B_\delta(x_{j,\varepsilon}^{(1)})}(|w_\varepsilon^{(1)}(x)|+|w_\varepsilon^{(2)}(x)|)\xi_\varepsilon(x)\mathrm{d}x\Big)+O(\varepsilon^\gamma).
\end{aligned} \tag{3.20}$$

Next, since $\psi_j(x)$ is an odd function with respect to x_j and an even function with respect to x_i for $i\neq j$, then using Proposition 2.7 and Lemma 3.2, we have

$$\begin{aligned}
&\int_{B_\delta(x_{j,\varepsilon}^{(1)})}|x-a_j|^{m-2}(x_i-a_{j,i})U_{a_j}\big(\frac{x-x_{j,\varepsilon}^{(1)}}{\varepsilon}\big)\xi_\varepsilon(x)\mathrm{d}x \\
&= \int_{B_\delta(x_{j,\varepsilon}^{(1)})}|x-a_j|^{m-2}(x_i-a_{j,i})U_{a_j}\big(\frac{x-x_{j,\varepsilon}^{(1)}}{\varepsilon}\big)\xi_\varepsilon(x)\mathrm{d}x \\
&= \varepsilon^{m+N-1}\Big(\int_{B_{\frac{\delta}{\varepsilon}}(0)}|x+\frac{x_{j,\varepsilon}^{(1)}-a_j}{\varepsilon}|^{m-2}(x_i+\frac{x_{j,\varepsilon,i}^{(1)}-a_{j,i}}{\varepsilon})U_{a_j}(x)\big(\sum_{l=1}^{N}b_{j,l}\psi_l(x)\big)\mathrm{d}x+o(1)\Big) \\
&= b_{j,i}\varepsilon^{m+N-1}\int_{\mathbb{R}^N}|x|^{m-2}x_iU_{a_j}(x)\psi_i(x)\mathrm{d}x+o\big(\varepsilon^{m+N-1}\big).
\end{aligned} \tag{3.21}$$

Also, (A.11) implies

$$\int_{B_\delta(x_{j,\varepsilon}^{(1)})} \big(|w_\varepsilon^{(1)}(x)| + |w_\varepsilon^{(2)}(x)|\big)\xi_\varepsilon(x)\mathrm{d}x = O\big((\|w_\varepsilon^{(1)}\|_\varepsilon + \|w_\varepsilon^{(2)}\|_\varepsilon)\|\xi_\varepsilon\|_\varepsilon\big) = O(\varepsilon^{m+N}). \tag{3.22}$$

Similar to (3.21) and (3.22), we can deduce

$$\int_{B_\delta(x_{j,\varepsilon}^{(1)})} |x-a_j|^{m-2}(x_i - a_{j,i})\nabla U_{a_j}\big(\frac{x - x_{j,\varepsilon}^{(1)}}{\varepsilon}\big)\xi_\varepsilon(x)\mathrm{d}x = O\big(\varepsilon^{m+N-1}\big), \tag{3.23}$$

and

$$\int_{B_\delta(x_{j,\varepsilon}^{(1)})} |x-a_j|^m \big(u_\varepsilon^{(1)}(x) + u_\varepsilon^{(2)}(x)\big)\xi_\varepsilon(x)\mathrm{d}x = O(\varepsilon^{m+N}). \tag{3.24}$$

Then, by choosing $\gamma > 0$ appropriately, from (3.18), (3.20), (3.21), (3.22), (3.23) and (3.24), we get

$$\begin{aligned}\int_{B_\delta(x_{j,\varepsilon}^{(1)})} &\frac{\partial V(x)}{\partial x_i}\big(u_\varepsilon^{(1)}(x) + u_\varepsilon^{(2)}(x)\big)\xi_\varepsilon(x)\mathrm{d}x \\ &= 2mb_i b_{j,i}\varepsilon^{m+N-1}\int_{\mathbb{R}^N} |x|^{m-2}x_i U_{a_j}(x)\psi_i(x)\mathrm{d}x + o\big(\varepsilon^{m+N-1}\big).\end{aligned} \tag{3.25}$$

So (3.14), (3.17) and (3.25) imply

$$2mb_i b_{j,i}\int_{\mathbb{R}^N} |x|^{m-2}x_i U_{a_j}(x)\psi_i(x)\mathrm{d}x = o(1).$$

This means $b_{j,i} = 0$. Similarly, we can obtain $b_{j,i} = 0$, for all $j = 1, \cdots, k$, $i = 1, \cdots, N$. □

Proposition 3.4. *For any fixed $R > 0$, it holds*

$$\xi_\varepsilon(x) = o(1),\ \ x \in \bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)}).$$

Proof. Lemma 3.2 and Lemma 3.3 show that for any fixed $R > 0$,

$$\xi_{\varepsilon,j}(x) = o(1) \text{ in } B_R(0),\ \ j = 1, \cdots, k.$$

Also, we know $\xi_{\varepsilon,j}(x) = \xi_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)})$, then $\xi_\varepsilon(x) = o(1), x \in B_{R\varepsilon}(x_{j,\varepsilon}^{(1)})$. □

Proposition 3.5. *For large $R > 0$, we have*

$$\xi_\varepsilon(x) = o(1),\ \ x \in \mathbb{R}^N \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)}).$$

Proof. First, we have

$$-\varepsilon^2\Delta\xi_\varepsilon(x)+V(x)\xi_\varepsilon(x)=C_\varepsilon(x)\xi_\varepsilon(x),$$

where

$$C_\varepsilon(x)=(p-1)\int_0^1\big(tu_\varepsilon^{(1)}(x)+(1-t)u_\varepsilon^{(2)}(x)\big)^{p-2}\mathrm{d}t.$$

Next, for $x\in\mathbb{R}^N\backslash\bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)})$, we know

$$|C_\varepsilon(x)|\le C\big(|w_\varepsilon^{(1)}(x)|^{p-2}+|w_\varepsilon^{(2)}(x)|^{p-2}\big)+o_R(1)+o_\varepsilon(1),\ \text{as } R\to\infty,\varepsilon\to 0. \tag{3.26}$$

Now, we estimate $w_\varepsilon^{(l)}(x)$ in $\mathbb{R}^N\backslash\bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)})$, $l=1,2$.

Setting $\tilde{w}_\varepsilon^{(l)}(x)=w_\varepsilon^{(l)}(\varepsilon x)$, then from (3.2), we obtain

$$-\Delta\tilde{w}_\varepsilon^{(l)}(x)+V(\varepsilon x)\tilde{w}_\varepsilon^{(l)}(x)=\tilde{N}(\tilde{w}_\varepsilon^{(l)}(x))+\tilde{l}_\varepsilon(\varepsilon x),\quad x\in\mathbb{R}^N\backslash\bigcup_{j=1}^k B_R(x_{j,\varepsilon}^{(1)}), \tag{3.27}$$

where

$$\begin{cases}\tilde{N}(\tilde{w}_\varepsilon^{(l)}(x))=\big(\sum\limits_{j=1}^k U_{a_j}(\frac{\varepsilon x-x_{j,\varepsilon}^{(l)}}{\varepsilon})+\tilde{w}_\varepsilon^{(l)}(x)\big)^{p-1}-\sum\limits_{j=1}^k U_{a_j}^{p-1}(\frac{\varepsilon x-x_{j,\varepsilon}^{(l)}}{\varepsilon}),\\ \tilde{l}_\varepsilon(\varepsilon x)=\sum\limits_{j=1}^k\big(V(\varepsilon a_j)-V(\varepsilon x)\big)U_{a_j}\big(\frac{\varepsilon x-x_{j,\varepsilon}^{(l)}}{\varepsilon}\big).\end{cases}$$

From (2.4) we deduce $\|\tilde{w}_\varepsilon^{(l)}\|_\varepsilon=o(1)$, and by the exponential decay of $U_{a_j}(\frac{\varepsilon x-x_{j,\varepsilon}^{(l)}}{\varepsilon})$ in W as $R\to\infty$, then

$$\|\tilde{w}_\varepsilon^{(l)}(x)\|_{H^1(W)}=o_\varepsilon(1)+o_R(1),\ \text{as}\ \ \varepsilon\to0,\ R\to\infty,$$

where $W=\mathbb{R}^N\backslash\bigcup_{j=1}^k B_R(x_{j,\varepsilon}^{(l)})$. So,

$$\|\tilde{N}(\tilde{w}_\varepsilon^{(l)}(x))\|_{L^{\frac{2^*}{p-1}}(W)}=o_\varepsilon(1),\ \text{and}\ \|\tilde{l}_\varepsilon(\varepsilon x)\|_{L^q(W)}=o_R(1),\ \forall q>1. \tag{3.28}$$

Combining (3.27), (3.28) and the L^p estimates, we have

$$\|\tilde{w}_\varepsilon^{(l)}(x)\|_{W^{2,\frac{2^*}{p-1}}(W)}=o_\varepsilon(1)+o_R(1).$$

Then, using the Sobolev embedding theorems and L^p estimates for finite steps, we obtain

$$\|\tilde{w}_\varepsilon^{(l)}(x)\|_{W^{2,q}(W)}=o_\varepsilon(1)+o_R(1),\ \text{for some}\ q\in(\frac{N}{2},N).$$

Next, using Sobolev embedding theorems again, we have

$$\|\tilde{w}_\varepsilon^{(l)}(x)\|_{L^\infty(W)}\le C\|\tilde{w}_\varepsilon^{(l)}(x)\|_{C^{0,2-\frac{N}{q}}(W)}=o_\varepsilon(1)+o_R(1).$$

This means

$$\|w_\varepsilon^{(l)}(x)\|_{L^\infty\left(\mathbb{R}^N\setminus\bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)})\right)} = o_\varepsilon(1) + o_R(1). \tag{3.29}$$

Then (3.26) and (3.29) show that for large R and small ε,

$$|C_\varepsilon(x)| \le \inf_{x\in\mathbb{R}^N} V(x).$$

Thus for large R, we have

$$\begin{cases} -\varepsilon^2\Delta\xi_\varepsilon + \big(V(x) - C_\varepsilon(x)\big)\xi_\varepsilon = 0, & x \in \mathbb{R}^N\setminus\bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)}),\\ \xi_\varepsilon(x) = o(1) \text{ (as } \varepsilon\to 0), & x \in \partial\big(\bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)})\big),\\ \xi_\varepsilon(x)\to 0, & \text{as } |x|\to 0, \end{cases}$$

and

$$V(x) - C_\varepsilon(x) \ge 0,\ x\in\mathbb{R}^N\setminus\bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)}).$$

By the maximum principle, we obtain

$$\xi_\varepsilon(x) = o(1),\ x\in\mathbb{R}^N\setminus\bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}^{(1)}).$$

□

Remark 3.6. *Since the nonlinear term of problem* (1.1) *is subcritical, we can not obtain the pointwise estimate of the error term $w_\varepsilon(x)$ by the similar methods in [18]. In our paper, we use the estimate of the norm $\|w_\varepsilon\|_\varepsilon$ to prove Proposition 3.4. On the other hand, in Proposition 3.5 we mainly use the technique of maximum principle.*

Proof of Theorem 1.1: Suppose that $u_\varepsilon^{(1)}(x)$, $u_\varepsilon^{(2)}(x)$ are two different positive solutions concentrating at k different points $\{a_1,\cdots,a_k\}$. From Proposition 3.4 and Proposition 3.5, for small ε, we have

$$\xi_\varepsilon(x) = o(1),\ x\in\mathbb{R}^N,$$

which is in contradiction with $\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 1$. So, $u_\varepsilon^{(1)}(x) \equiv u_\varepsilon^{(2)}(x)$ for small ε. (1.7) follows from Proposition 2.1, Proposition 2.6 and Proposition 2.7. □

Appendix

In this appendix, we give various estimates and results which have been used repeatedly in previous sections.

A. Some Estimates

First, from the exponential decay of $U_{a_j}(x)$ for $j = 1, \cdots, k$, we have

Lemma A.1. *There exists a small constant d_1, such that, for any $\gamma > 0$, we have*

$$U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) = O(\varepsilon^\gamma), \text{ for } x \in B_d(x_{i,\varepsilon}), \ j \neq i \text{ and } 0 < d < d_1, \tag{A.1}$$

and

$$U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) = O(\varepsilon^\gamma), \text{ for } x \in \partial B_d(x_{i,\varepsilon}), \ i = 1, \cdots, k \text{ and } 0 < d < d_1. \tag{A.2}$$

Lemma A.2. *For any $\gamma > 0$, it holds*

$$\int_{\mathbb{R}^N} U_{a_i}^{q_1}(\frac{x - x_{i,\varepsilon}}{\varepsilon}) U_{a_j}^{q_2}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) \mathrm{d}x = O(\varepsilon^\gamma), \tag{A.3}$$

and

$$\int_{\mathbb{R}^N} \varepsilon^2 \nabla U_{a_i}(\frac{x - x_{i,\varepsilon}}{\varepsilon}) \nabla U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) \mathrm{d}x = O(\varepsilon^\gamma), \tag{A.4}$$

where $i, j = 1, \cdots, k$, $i \neq j$ and $q_1, q_2 > 0$.

Proof. Taking a small constant d and using (A.1), for any $\gamma > 0$, we have

$$\int_{B_d(x_{i,\varepsilon})} U_{a_i}^{q_1}(\frac{x - x_{i,\varepsilon}}{\varepsilon}) U_{a_j}^{q_2}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) \mathrm{d}x \leq C\varepsilon^\gamma \int_{B_d(x_{i,\varepsilon})} U_{a_i}^{q_1}(\frac{x - x_{i,\varepsilon}}{\varepsilon}) \mathrm{d}x \leq C\varepsilon^\gamma.$$

Similarly,

$$\int_{B_d(x_{j,\varepsilon})} U_{a_i}^{q_1}(\frac{x - x_{i,\varepsilon}}{\varepsilon}) U_{a_j}^{q_2}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) \mathrm{d}x \leq C\varepsilon^\gamma,$$

and

$$\int_{\mathbb{R}^N \setminus \left(B_d(x_{i,\varepsilon}) \bigcup B_d(x_{j,\varepsilon})\right)} U_{a_i}^{q_1}(\frac{x - x_{i,\varepsilon}}{\varepsilon}) U_{a_j}^{q_2}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) \mathrm{d}x \leq C\varepsilon^\gamma.$$

The above inequalities imply (A.3). Also combining (1.4), (A.2) and the proof of (A.3), we know (A.4). □

Lemma A.3. *Suppose that $V(x)$ satisfies* (1.5)*, then we have*

$$\int_{\mathbb{R}^N} \big(V(a_j) - V(x)\big) U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) u(x) \mathrm{d}x = O\big(\varepsilon^{\frac{N}{2}+m} + \varepsilon^{\frac{N}{2}} \max_{j=1,\cdots,k.} |x_{j,\varepsilon} - a_j|^m\big) \|u\|_\varepsilon, \tag{A.5}$$

and

$$\int_{\mathbb{R}^N} \frac{\partial V(x)}{\partial x_i} U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) u(x) \mathrm{d}x = O\big(\varepsilon^{\frac{N}{2}+m-1} + \varepsilon^{\frac{N}{2}} \max_{j=1,\cdots,k.} |x_{j,\varepsilon} - a_j|^{m-1}\big) \|u\|_\varepsilon, \tag{A.6}$$

where $u(x) \in H_\varepsilon$ and $j = 1, \cdots, k$.

Proof. First, from (1.5) and Hölder's inequality, for a small constant d, we have

$$\begin{aligned}
&\Big|\int_{B_d(x_{j,\varepsilon})}\big(V(a_j)-V(x)\big)U_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)u(x)\mathrm{d}x\Big| \\
&\qquad\le C\int_{B_d(x_{j,\varepsilon})}|x-a_j|^m U_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)|u(x)|\mathrm{d}x \\
&\qquad\le C\Big(\int_{B_d(x_{j,\varepsilon})}|x-a_j|^{2m}U^2_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_d(x_{j,\varepsilon})}u^2(x)\mathrm{d}x\Big)^{\frac12} \qquad (A.7)\\
&\qquad\le C\varepsilon^{\frac N2}\Big(\int_{B_{\frac{d}{\varepsilon}}(0)}|\varepsilon y+(x_{j,\varepsilon}-a_j)|^{2m}U^2_{a_j}(y)\mathrm{d}y\Big)^{\frac12}\|u\|_\varepsilon \\
&\qquad\le C\varepsilon^{\frac N2}\big(\varepsilon^m+|x_{j,\varepsilon}-a_j|^m\big)\|u\|_\varepsilon.
\end{aligned}$$

Also, by the exponential decay of $U_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)$ in $\mathbb{R}^N\backslash B_d(x_{j,\varepsilon})$, we can deduce that, for any $\gamma>0$,

$$\Big|\int_{\mathbb{R}^N\backslash B_d(x_{j,\varepsilon})}\big(V(a_j)-V(x)\big)U_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)u(x)\mathrm{d}x\Big|\le C\varepsilon^\gamma\|u\|_\varepsilon. \qquad (A.8)$$

Then, taking suitable $\gamma>0$, from (A.7) and (A.8), we get (A.5).

Next, similar to (A.7) and (A.8), for any $\gamma>0$, we have

$$\begin{aligned}
\Big|\int_{B_d(x_{j,\varepsilon})}\frac{\partial V(x)}{\partial x_i}U_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)u(x)\mathrm{d}x\Big| &\le C\int_{B_d(x_{j,\varepsilon})}|x-a_j|^{m-1}U_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)|u(x)|\mathrm{d}x \qquad (A.9)\\
&\le C\varepsilon^{\frac N2}\big(\varepsilon^{m-1}+|x_{j,\varepsilon}-a_j|^{m-1}\big)\|u\|_\varepsilon,
\end{aligned}$$

and

$$\Big|\int_{\mathbb{R}^N\backslash B_d(x_{j,\varepsilon})}\frac{\partial V(x)}{\partial x_i}U_{a_j}\big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\big)u(x)\mathrm{d}x\Big|\le C\varepsilon^\gamma\|u\|_\varepsilon. \qquad (A.10)$$

Then, taking suitable $\gamma>0$, (A.9) and (A.10) imply (A.6). □

Lemma A.4. *Suppose that $u(x),v(x)\in H_\varepsilon$ and $\|u\|_\varepsilon=o(\varepsilon^{\frac N2})$, then it holds*

$$\int_{\mathbb{R}^N}u^{p-1}(x)v(x)\mathrm{d}x=o\big(\|u\|_\varepsilon\|v\|_\varepsilon\big),\ \text{for any } 2<p<\frac{2N}{N-2}. \qquad (A.11)$$

Proof. First, for any $s\in(2,\frac{2N}{N-2})$, using Hölder's inequality and the fact that $\|u\|_\varepsilon=o(\varepsilon^{\frac N2})$, we can deduce

$$\begin{aligned}
\int_{\mathbb{R}^N}|u(x)|^s\mathrm{d}x &\le C\Big(\int_{\mathbb{R}^N}|u(x)|^{2^*}\mathrm{d}x\Big)^{\frac{(N-2)(s-2)}{4}}\cdot\Big(\int_{\mathbb{R}^N}|u(x)|^2\mathrm{d}x\Big)^{\frac{2N-p(N-2)}{4}} \\
&\le C\big(\varepsilon^{-1}\|u\|_\varepsilon\big)^{\frac{N(s-2)}{2}}\cdot\|u\|_\varepsilon^{\frac{2N-s(N-2)}{2}} \qquad (A.12)\\
&\le C\varepsilon^{-\frac{N(s-2)}{2}}\|u\|^s_\varepsilon=o\big(\|u\|^2_\varepsilon\big).
\end{aligned}$$

Next, by Hölder's inequality and Sobolev inequality, we see

$$\Big|\int_{\mathbb{R}^N} u^{p-1}(x)v(x)\mathrm{d}x\Big| \leq \begin{cases} \big(\int_{\mathbb{R}^N} |u(x)|^{2(p-1)}\mathrm{d}x\big)^{\frac{1}{2}}\|v\|_\varepsilon, & 2 < p < \frac{2N-2}{N-2}; \\ \big(\int_{\mathbb{R}^N} |u(x)|^{\frac{2Np}{N+2}}\mathrm{d}x\big)^{\frac{N+2}{2N}} \cdot \big(\varepsilon^{-1}\|v\|_\varepsilon\big), & \frac{2N-2}{N-2} \leq p < \frac{2N}{N-2}. \end{cases} \tag{A.13}$$

Then from $\|u\|_\varepsilon = o(\varepsilon^{\frac{N}{2}})$, (A.12) and (A.13), we get (A.11). □

Lemma A.5. *For any fixed number $l \in \mathbb{N}^+$, suppose that $\{u_i(x)\}_{i=1}^l$ satisfies*

$$\int_{\mathbb{R}^N} |u_i(x)|\mathrm{d}x < +\infty,\ i = 1, \cdots, l.$$

Then for any x_0, there exist a small constant d and another constant C such that

$$\int_{\partial B_d(x_0)} |u_i(x)|\mathrm{d}\sigma \leq C\int_{\mathbb{R}^N} |u_i(x)|\mathrm{d}x,\ \textit{for all } i = 1, \cdots, l. \tag{A.14}$$

Proof. Let $M_i = \int_{\mathbb{R}^N} |u_i(x)|\mathrm{d}x$, for $i = 1, \cdots, l$. Then for a fixed small $r_0 > 0$,

$$\int_{B_{r_0}(x_0)} \big(\sum_{i=1}^l |u_i(x)|\big)\mathrm{d}x \leq \sum_{i=1}^l M_i,\ \text{for all } i = 1, \cdots, l. \tag{A.15}$$

On the other hand,

$$\int_{B_{r_0}(x_0)} \big(\sum_{i=1}^l |u_i(x)|\big)\mathrm{d}x \geq \int_0^{r_0}\int_{\partial B_r(x_0)} \big(\sum_{i=1}^l |u_i(x)|\big)\mathrm{d}\sigma dr. \tag{A.16}$$

Then (A.15) and (A.16) imply that there exists a constant $d < r_0$ such that

$$\int_{\partial B_r(x_0)} |u_i(x)|\mathrm{d}\sigma \leq \int_{\partial B_d(x_0)} \big(\sum_{i=1}^l |u_i(x)|\big)\mathrm{d}\sigma \leq \frac{\sum_{i=1}^l M_i}{r_0},\ \text{for all } i = 1, \cdots, l. \tag{A.17}$$

So taking $C = \max\limits_{1\leq i\leq l} \dfrac{\sum_{i=1}^l M_i}{r_0 M_i}$, we can obtain (A.14) from (A.17). □

B. Analysis of w_ε and v_ε

By Proposition 2.1, let

$$u_\varepsilon(x) = \sum_{j=1}^k U_{a_j}\big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\big) + w_\varepsilon(x) = \sum_{j=1}^k (1 + \alpha_{j,\varepsilon})U_{a_j}\big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\big) + v_\varepsilon(x)$$

be the solution of (1.1), then we have

$$-\varepsilon^2\Delta w_\varepsilon(x)+V(x)w_\varepsilon(x)-(p-1)\sum_{j=1}^k U_{a_j}^{p-2}(\frac{x-x_{j,\varepsilon}}{\varepsilon})w_\varepsilon(x)=N\big(w_\varepsilon(x)\big)+l_\varepsilon(x), \tag{B.1}$$

where

$$\begin{cases} N\big(w_\varepsilon(x)\big)=\big(\sum_{j=1}^k U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon})+w_\varepsilon(x)\big)^{p-1}-\sum_{j=1}^k U_{a_j}^{p-1}(\frac{x-x_{j,\varepsilon}}{\varepsilon}) \\ \qquad\qquad -(p-1)\sum_{j=1}^k U_{a_j}^{p-2}(\frac{x-x_{j,\varepsilon}}{\varepsilon})w_\varepsilon(x), \\ l_\varepsilon(x)=\sum_{j=1}^k (V(a_j)-V(x))U_{a_j}(\frac{x-x_{j,\varepsilon}}{\varepsilon}). \end{cases}$$

Lemma B.1. *There exists a constant $\rho>0$ independent of ε such that*

$$\int_{\mathbb{R}^N}\big[\varepsilon^2|\nabla u(x)|^2+V(x)u^2(x)-(p-1)\sum_{j=1}^k U_{a_j}^{p-2}(\frac{x-x_{j,\varepsilon}}{\varepsilon})u^2(x)\big]\mathrm{d}x\geq\rho\|u\|_\varepsilon^2, \tag{B.2}$$

for all $u(x)\in\bigcap_{j=1}^k E_{\varepsilon,a_j,x_{j,\varepsilon}}$ and $x_{j,\varepsilon}\in B_\delta(a_j)$ provided $\delta>0$ is sufficiently small.

Proof. See [[10], Proposition C.1]. □

Proposition B.2. *It holds*

$$\|w_\varepsilon\|_\varepsilon^2=O(\varepsilon^{N+2m})+O\big(\varepsilon^N\max_{j=1,\cdots,k}|x_{j,\varepsilon}-a_j|^{2m}\big)+O(\varepsilon^N\sum_{j=1}^k\alpha_{j,\varepsilon}^2), \tag{B.3}$$

and

$$\|\upsilon_\varepsilon\|_\varepsilon^2=O(\varepsilon^{N+2m})+O\big(\varepsilon^N\max_{j=1,\cdots,k}|x_{j,\varepsilon}-a_j|^{2m}\big)+O(\varepsilon^N\sum_{j=1}^k\alpha_{j,\varepsilon}^2). \tag{B.4}$$

Proof. First, from Lemma B.1, we know

$$\begin{aligned}\|\upsilon_\varepsilon\|_\varepsilon^2&\leq C\big(\int_{\mathbb{R}^N}\big[\varepsilon^2|\nabla\upsilon_\varepsilon(x)|^2+V(x)\upsilon_\varepsilon^2(x)-(p-1)\sum_{j=1}^k U_{a_j}^{p-2}(\frac{x-x_{j,\varepsilon}}{\varepsilon})\upsilon_\varepsilon^2(x)\big]\mathrm{d}x\big)\\ &=C\big(\int_{\mathbb{R}^N}\big[\varepsilon^2|\nabla w_\varepsilon(x)|^2+V(x)w_\varepsilon^2(x)-(p-1)\sum_{j=1}^k U_{a_j}^{p-2}(\frac{x-x_{j,\varepsilon}}{\varepsilon})w_\varepsilon^2(x)\big]\mathrm{d}x\big)\\ &\quad+C\big(A_1+A_2\big),\end{aligned} \tag{B.5}$$

where

$$A_1 = -\|\sum_{j=1}^{k} \alpha_{j,\varepsilon} U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon})\|_\varepsilon + (p-1) \sum_{l=1}^{k} \int_{\mathbb{R}^N} U_{a_l}^{p-2}(\frac{x - x_{l,\varepsilon}}{\varepsilon})\big(\sum_{j=1}^{k} \alpha_{j,\varepsilon} U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon})\big)^2 \mathrm{d}x,$$

and

$$\begin{aligned} A_2 = {} & 2(p-1) \sum_{j=1}^{k} \sum_{l=1}^{k} \alpha_{j,\varepsilon} \int_{\mathbb{R}^N} U_{a_l}^{p-2}(\frac{x - x_{l,\varepsilon}}{\varepsilon}) U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) \upsilon_\varepsilon(x) \mathrm{d}x \\ & - 2 \sum_{j=1}^{k} \alpha_{j,\varepsilon} \big(U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}), \upsilon_\varepsilon(x)\big)_\varepsilon. \end{aligned}$$

From (A.3) and (A.4), for any $\gamma > 0$, we know

$$A_1 = O\big(\sum_{j=1}^{k} \alpha_{j,\varepsilon}^2 \varepsilon^N\big) + O(\varepsilon^\gamma). \tag{B.6}$$

Then $\upsilon_\varepsilon(x) \in \bigcap_{j=1}^{k} E_{\varepsilon, a_j, x_{j,\varepsilon}}$, (1.3) and (A.3) imply that, for any $\gamma > 0$,

$$\begin{aligned} A_2 &= 2(p-1) \sum_{j=1}^{k} \alpha_{j,\varepsilon} \int_{\mathbb{R}^N} U_{a_j}^{p-1}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) \upsilon_\varepsilon(x) \mathrm{d}x + O(\varepsilon^\gamma) \\ &= 2(p-1) \sum_{j=1}^{k} \alpha_{j,\varepsilon} \int_{\mathbb{R}^N} \big(V(a_j) - V(x)\big) U_{a_j}\big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\big) \upsilon_\varepsilon(x) \mathrm{d}x + O(\varepsilon^\gamma). \end{aligned} \tag{B.7}$$

Then (A.5) and (B.7) imply

$$A_2 = O\big(\sum_{j=1}^{k} \alpha_{j,\varepsilon}(\varepsilon^{\frac{N}{2}+m} + \varepsilon^{\frac{N}{2}} \max_{j=1,\cdots,k} |x_{j,\varepsilon} - a_j|^m)\big) \|\upsilon_\varepsilon\|_\varepsilon + O(\varepsilon^\gamma). \tag{B.8}$$

So, combining (B.5), (B.6) and (B.8), for any $\gamma > 0$, we have

$$\|\upsilon_\varepsilon\|_\varepsilon^2 = O\big(\int_{\mathbb{R}^N} \big[\varepsilon^2 |\nabla w_\varepsilon(x)|^2 + V(x) w_\varepsilon^2(x) - (p-1) W_{x,\varepsilon}^{p-2} w_\varepsilon^2(x)\big] \mathrm{d}x + \sum_{j=1}^{k} \alpha_{j,\varepsilon}^2 \varepsilon^N + \varepsilon^\gamma\big). \tag{B.9}$$

Also $\upsilon_\varepsilon(x) \in \bigcap_{j=1}^{k} E_{\varepsilon, a_j, x_{j,\varepsilon}}$, Lemma B.1 and (B.9) imply that, for any $\gamma > 0$,

$$\begin{aligned} \|w_\varepsilon\|_\varepsilon^2 &= \|\upsilon_\varepsilon\|_\varepsilon^2 + \sum_{j=1}^{k} \|\alpha_{j,\varepsilon} U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon})\|_\varepsilon^2 + 2\big(\upsilon_\varepsilon(x), \sum_{j=1}^{k} \alpha_{j,\varepsilon} U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon})\big)_\varepsilon + I_1 \\ &\le C \int_{\mathbb{R}^N} \big[\varepsilon^2 |\nabla w_\varepsilon(x)|^2 + V(x) w_\varepsilon^2(x) - (p-1) \sum_{j=1}^{k} U_{a_j}^{p-2}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) w_\varepsilon^2(x)\big] \mathrm{d}x \\ &\quad + C\big(\varepsilon^\gamma + \sum_{j=1}^{k} \alpha_{j,\varepsilon}^2 \varepsilon^N\big) + A_3, \end{aligned} \tag{B.10}$$

where

$$A_3 = \sum_{j=1}^{k} \sum_{i \neq j} \int_{\mathbb{R}^N} \alpha_{j,\varepsilon} \alpha_{i,\varepsilon} \big(U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}), U_{a_i}(\frac{x - x_{i,\varepsilon}}{\varepsilon}) \big)_\varepsilon .$$

Also, (A.3) implies that, for any $\gamma > 0$,

$$A_3 = O(\varepsilon^\gamma). \tag{B.11}$$

Then (2.2), (A.3), (A.4), (B.10) and (B.11) imply, for any $\gamma > 0$,

$$\begin{aligned} \|w_\varepsilon\|_\varepsilon^2 = & O\big(\int_{\mathbb{R}^N} \big[\varepsilon^2 |\nabla w_\varepsilon(x)|^2 + V(x) w_\varepsilon^2(x) - (p-1) \sum_{j=1}^{k} U_{a_j}^{p-2}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) w_\varepsilon^2(x) \big] \mathrm{d}x \big) \\ & + O(\sum_{j=1}^{k} \alpha_{j,\varepsilon}^2 \varepsilon^N + \varepsilon^\gamma). \end{aligned} \tag{B.12}$$

On the other hand, from (B.1), we know

$$\begin{aligned} & \int_{\mathbb{R}^N} \big[\varepsilon^2 |\nabla w_\varepsilon(x)|^2 + V(x) w_\varepsilon^2(x) - (p-1) \sum_{j=1}^{k} U_{a_j}^{p-2}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) w_\varepsilon^2(x) \big] \mathrm{d}x \\ & = \int_{\mathbb{R}^N} \big[N\big(w_\varepsilon(x)\big) w_\varepsilon(x) + l_\varepsilon(x) w_\varepsilon(x) \big] \mathrm{d}x. \end{aligned} \tag{B.13}$$

Then combining (2.4), (A.3) and (A.11), we can obtain

$$\int_{\mathbb{R}^N} N\big(w_\varepsilon(x)\big) w_\varepsilon(x) \mathrm{d}x \le C\big(\int_{\mathbb{R}^N} |w_\varepsilon(x)|^{p^*} \mathrm{d}x + \varepsilon^\gamma \big) = o(1) \|w_\varepsilon\|_\varepsilon^2 + O(\varepsilon^\gamma), \tag{B.14}$$

where $p^* = \min\{p, 3\}$. Also, (A.5) imply that

$$\int_{\mathbb{R}^N} l_\varepsilon(x) w_\varepsilon(x) \mathrm{d}x = O\big(\varepsilon^{\frac{N}{2} + m} + \varepsilon^{\frac{N}{2}} \max_{j=1,\cdots,k.} |x_{j,\varepsilon} - a_j|^m \big) \|w_\varepsilon\|_\varepsilon. \tag{B.15}$$

Then taking suitable $\gamma > 0$, by (B.12), (B.13), (B.14) and (B.15), we get (B.3). Also, (B.3), (B.12) and (B.13) imply (B.4). $\square$

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